

## Lecture 14: Tree

**Definition 1** Let  $G$  be a graph. A vertex  $v$  of  $G$  is called a cut-vertex if  $G - v$  has more components than  $G$ .

**Theorem 1** Let  $G$  be a connected graph with  $|G| \geq 2$  and let  $v \in V(G)$ .

*a: If  $\deg(v) = 1$ , then  $G - v$  is connected, so that  $v$  is never a cut-vertex.*

*b: If  $G - v$  is connected, then either  $\deg(v) = 1$  or  $v$  is on a cycle.*

**Proof:** (a) Let  $a, b \in V(G - v)$ ,  $a \neq b$ . Since  $G$  is connected there is a  $a - b$  path in  $G$ . Evidently the vertex  $v$  can not be the internal vertex of this path, as the degree of internal vertex is  $\geq 2$ . So the path  $a - b$  is available in  $G - v$ . So,  $G - v$  is connected.

(b) Assume  $G - v$  is connected. If  $\deg(v) = 1$ , then nothing to prove. So let  $\deg(v) \geq 2$ . To show that  $v$  is on a cycle in  $G$ . Let  $u$  and  $w$  be two distinct neighbors of  $v$ . Since  $G - v$  is connected, there is a path  $(u = u_1, u_2, \dots, u_k = w)$  in  $G - v$ . Then  $(u = u_1, u_2, \dots, u_k = w, v)$  is a cycle.

**Definition 2** Let  $G$  be a graph. An edge  $e$  in  $G$  is called a cut-edge or a bridge if  $G - e$  has more connected components than that of  $G$ .

**Proposition 1** Let  $G$  be connected and let  $e = uv$  be a cut-edge. Then  $G - e$  has two components, one containing  $u$  and the other containing  $v$ .

**Proof:** If  $G - e$  is not disconnected, then by definition,  $e$  can not be a cut edge. So  $G - e$  has at least two components. Let  $G_u$  (respectively,  $G_v$ ) be the component containing the vertex  $u$  (respectively,  $v$ ). We claim that these are the only components.

Let  $w \in V(G)$ . Since  $G$  is connected, there is a path, say  $P$ , from  $w$  to  $u$ . Moreover, either  $P$  contains  $v$  as its internal vertex or  $P$  does not contain  $v$ . In the first case,  $w \in V(G_v)$  and in the latter case,  $w \in V(G_u)$ . Thus, every vertex of  $G$  is either in  $V(G_v)$  or in  $V(G_u)$  and hence the required result follows.

**Theorem 2** Let  $G$  be a graph and let  $e$  be an edge. Then,  $e$  is a cut-edge iff  $e$  is not on a cycle.

**Proof:** Suppose  $e = uv$  is a cut-edge of  $G$ . Let  $F$  be the component of  $G$  that contains  $e$ . Then, by the above Proposition,  $F - e$  has two components, namely,  $F_u$  that contains  $u$  and  $F_v$  that contains  $v$ .

Let if possible,  $C = (u, v = v_1, \dots, v_k = u)$  be a cycle containing  $e = uv$ . Then  $(v = v_1, \dots, v_k = u)$  is a  $u - v$  path in  $F - e$ . Hence,  $F - e$  is still connected, a contradiction. Thus,  $e$  cannot be on any cycle.

Conversely, let  $e = uv$  be an edge which is not on any cycle. Now, suppose that  $F$  is the component of  $G$  that contains  $e$ . We need to show that  $F - e$  is disconnected. Let if possible, there is a  $u - v$  path, say  $(u = u_1, \dots, u_k = v)$ , in  $F - e$ . Then,  $(v, u = u_1, \dots, u_k = v)$  is a cycle containing  $e$ . A contradiction to  $e$  not lying on any cycle.

**Definition 3** A connected graph  $G$  with no cycles is called a tree. A collection of trees is called a forest.

**Proposition 2** A tree on  $n$  vertices has  $n - 1$  edges.

**Proof:** We apply induction on  $n$ . Take a tree on  $n \geq 2$  vertices and delete an edge  $e$ . Then, we get two subtrees  $T_1, T_2$  of order  $n_1, n_2$ , respectively, where  $n_1 + n_2 = n$ . So,  $E(T) = E(T_1) \cup E(T_2) \cup \{e\}$ . By induction hypothesis  $|E(T)| = |E(T_1)| + |E(T_2)| + 1 = n_1 - 1 + n_2 - 1 + 1 = n_1 + n_2 - 1 = n - 1$ .

**Corollary 1** A tree with at least two vertices has at least two pendant vertices.

**Proof:** Let  $T$  be a tree on  $n \geq 2$  vertices. Then  $\sum_{v \in V(T)} \deg(v) = 2|E(T)| = 2n - 2$ . Suppose  $T$  has  $k$  pendent vertices. Then the remaining vertices, except these  $k$  vertices, have degree  $\geq 2$ . Then by Handshaking Lemma,  $\sum_{v \in V(T)} \deg(v) \geq 2k + 2(n - k) = 2n - k$ . Then by the above expression of degree sum,  $k \geq 2$ .

We now prove that the following statements that characterize trees are equivalent.

**Theorem 3** Let  $G = (V, E)$  be a graph with  $|V| = n$  and  $|E| = m$ . Then f.s.a.e.

1.  $G$  is a tree.
2. Let  $u, v \in V$ . Then there is a unique path from  $u$  to  $v$ .
3.  $G$  is connected and  $n = m + 1$ .

**Proof:** (1  $\Rightarrow$  2): Since  $G$  is connected, for each  $u, v \in V$ , there is a path from  $u$  to  $v$ . On the contrary, let us assume that there are two distinct  $u - v$  paths  $P_1$  and  $P_2$ . Then by earlier result, the subgraph  $P_1 \cup P_2$  contains a cycle. This contradicts the assumption that  $G$  is a tree (it has no cycle).

(2  $\Rightarrow$  3): We prove this by induction on the number of vertices of a graph. The result is clearly true for  $n = 1$  or  $n = 2$ . Let the result be true for all graphs that have  $n$  or less than  $n$  vertices. Now, consider a graph  $G$  on  $n + 1$  vertices that satisfies the condition 2. The uniqueness of the path implies that if we remove an edge, say  $e \in E$ , then the graph  $G$  will become disconnected. That is,  $G \setminus e$  will have exactly two components. Let the number of vertices in the two components be  $n_1$  and  $n_2$ . Then  $n_1, n_2 \leq n$  and  $n_1 + n_2 = n + 1$ . Hence, by induction hypothesis, the number of edges in  $G - e$  equals

$(n_1 - 1) + (n_2 - 1) = n_1 + n_2 - 2 = n - 1$  and hence the number of edges in  $G$  equals  $n - 1 + 1 = n$ . Thus, by the principle of mathematical induction, the result holds for all graphs that have a unique path from each pair of vertices.

(3  $\Rightarrow$  1): It is already given that  $G$  is a connected graph. We need to show that  $G$  has no cycle. So, on the contrary, let us assume that  $G$  has a cycle of length  $k$ . Then this cycle has  $k$  vertices and  $k$  edges. Now, consider the  $n - k$  vertices that do not lie of the cycle. Then for each vertex (corresponding to the  $n - k$  vertices), there will be a distinct edge incident with it on the smallest path from the vertex to the cycle. Hence, the number of edges will be greater than or equal to  $k + (n - k) = n$ . A contradiction to the assumption that the number of edges equals  $n - 1$ . Thus, the required result follows.